Numerical computation of the Brinkman system in a heterogeneous porous medium by mini-element $P_1 - Bubble/P_1$

1st EL OUADEFLI Lahcen
Mechanical Engineering Laboratory (LGM)
Sciences and Technologies Faculty (FST)
Fès, Morocco
lahcen.elouadefli@usmba.ac.ma

2nd EL AKKAD Abdeslam
Département de mathématiques,
Centre Régional des Métiers d'Education et de Formation (CRMEF)
Fès Meknès, Morocco
elakkadabdeslam1@gmail.com

3rd EL MOUTEA Omar
Laboratoire des Mathématiques
& Applications "LMA"
École Normale Supérieure–ENS.
Casablanca, Morocco
omar.elmoutea-etu@etu.univh2c.ma

4th ELKHALFI Ahmed
Mechanical Engineering Laboratory (LGM)
Sciences and Technologies Faculty (FST)
Fès, Morocco
aelkhalfi@gmail.com

5th CHAKER Hamza
Departement of engineering
Higher Institute Of Engineering
and Business, (ISGA)
Fez, Morocco
hamza.chaker@isga.ma

***Abstract*—This paper is interested in a numerical method, we use the mini-element $P_1 - Bubble/P_1$ over triangles, as a solver to the steady Brinkman flow equation with the Dirichlet boundary condition in a heterogeneous porous media. We define the necessary hypotheses to prove the existence and uniqueness of the solution. An iterative solver for the global linear system (Uzawa conjugate gradient method) is applied to accelerate the approach solution. A series of numerical examples with *Matlab software* demonstrates the effectiveness of this method for these equations arising in modeling flow in anisotropic porous media.**

***Index Terms*—Anisotropic porous media, Brinkman equation, Uzawa algorithm, Mini-element $P_1 - Bubble/P_1$,**

I. Introduction

In this paper, we propose to simulate the Brinkman equation by the finite element methods, we use the mini-element $P1 - Bubble/P1$ (see [3, 4, 15, 16, 17]) in $2D$. This element for spatial discretization of the different mathematcal equations is easy to use, practice since it allows for the use of equal-order interpolation (the same mesh for velocity and pressure). Equal-order interpolation is very useful in large-scale multi-physics codes to solve different problems applied in different domains (geology, engineering, ...). The Brinkman model describes the viscous fluid creeping flow in a highly heterogeneous porous medium [21]. This model has been studied by some researchers. In [22] the authors propose new nonconforming robust finite element spaces for the unknown variables (velocity, pressure) and detailed argument for the case of perturbation solution. Priori and a posteriori error estimates to ensure the mini-element to be uniformly stable are studies in [25]. Other more methods, such as the Galerkin finite element method in [24] and the multigrid method [23] are applied to the Brinkman model without considering parameter dependence of solution. *A. Ern* studies by finite element method (FEM) different equations [17] and *F. Brezzi, M. Fortin* for mixed Hybrid Finite Element Methods (MFEM, HFEM) see [15]. *V. Giraut, P. A. Raviart* uses the finite element method to solve one of the important nonlinear problems Navier-Stokes Equations [16].

The plan of this paper is as follows: In Section 2 we state by define the model problem and hypothesis to proof the existence and uniqueness of the fort and weak solution. The next section 3 we introduce the mini-element approximation $P_1 - Bubble/P_1$ to the Brikman system. The section 4 is devoted to the algebraic system to this problem, since the system has a big matrix we use "Uzawa conjugate algorithm" to accelerate the convergence of the solution. Numerical results are discussed in Section 5.

II. The model problem

The Brinkman problem describes flow through a porous media, and it can also be seen as a singular perturbation problem. We can use this equation modeling flow fluid in highly heterogeneous porous media see [19, 20]. Here we use it to describe viscous flow fluid in a heterogeneous porous media.

Let be $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$ a bounded open set with Lipschitz boundary $\partial\Omega$. The Brinkman system for heterogeneous porous media is represented by the following equations:

$$\begin{cases} -\nabla \cdot (\tilde{\mu}\nabla u) + \nabla p + \mu \mathbf{K}^{-1} u = f & in \quad \Omega, \\ \nabla \cdot u = 0 & in \quad \Omega, \end{cases} \tag{1}$$

978-1-6654-4238-1/21/$31.00 ©2021 IEEE

with the Dirichlet boundary condition

$$u = 0 \quad on \quad \partial\Omega. \tag{2}$$

The vector u represents the velocity field, p the pressure defined in the space $L^2(\Omega)$ and satisfy $\int_\Omega p \quad dx = 0$. We assume that f the external volumetric force acting on the fluid ($f \in [L^2(\Omega)]^d$). The function $\tilde{\mu}$ is the Newtonian fluid vescosity and the function μ is the physical dynamic viscosity that defines the fluid under consideration (e.g., water, oil, etc.). The vescosity $\tilde{\mu}$ and μ are continuous bounded functions. The matrix $\mathbf{K}$ define the permeability of the reservoir, the role of permeability is to controls the directional movement and the flow fluids rate of the reservoir in the formation and represents reciprocal of residence which porous medium offers to fluid flow. The K^{-1} is a tensor such that, there exist a two constants $b_1, b_2 > 0$ checking:

$$b_1\varphi^t\varphi \leq \varphi^t K^{-1}\varphi \leq b_2\varphi^t\varphi, \qquad \varphi \in \mathbb{R}^d. \tag{3}$$

and we assume that $g \in [L^2(\partial\Omega)]^d$. Let introduce the spaces:

$$V = [H_0^1(\Omega)]^d, \tag{4}$$

for the velocity field and

$$Q = \{q \in L^2(\Omega) / \int_\Omega q \quad dx = 0\}. \tag{5}$$

for the pressure equation. It's simple to see that: the Brinkman problem (1) has a unique solution $(u, p) \in V \times Q$ (see [18, lemma 2.1]). To study the numerical solution of this problem by finite element methods we will be first define the weak formulation of the problem.
The weak formulation of the system (1) is to find $(u, p) \in V \times Q$ such that:

$$\begin{cases} a(u, v) + b(v, p) = F(v) & \forall v \in V, \\ b(q, u) = 0 & \forall q \in Q, \end{cases} \tag{6}$$

where the bilinear form $a(\cdot,\cdot)$ is positive continuous coercive defined by:

$$a(u, v) = \int_\Omega \tilde{\mu}\nabla u \nabla v \; dx + \int_\Omega K^{-1}\mu u.v \; dx, \tag{7}$$

the bilinear form $b(\cdot,\cdot)$ is continuous and satisfies the $inf-sup$ condition (i.e there exists a constant $\beta > 0$ such that, $\sup_{v\in V} \frac{b(q,v)}{\|v\|_V} \geq \beta \parallel q \parallel_Q, \; \forall q \in Q$) defined by:

$$b(v, p) = -\int_\Omega p\nabla \cdot v \quad dx, \tag{8}$$

and the second member $F(\cdot)$ is a linear continuous function defined by:

$$F(v) = \int_\Omega f \cdot v \quad dx. \tag{9}$$

Now, we provide the spaces V and Q with the following norms:

$$\parallel u \parallel_V = a(u, u)^{\frac{1}{2}}, \quad \forall u \in V, \tag{10}$$

and

$$\parallel q \parallel_Q = \left(\int_\Omega \mid q \mid^2 dx\right)^{\frac{1}{2}}, \; \forall q \in Q. \tag{11}$$

It's simple to prove that the norm $\| \cdot \|_V$ is equivalent to $\| \cdot \|_{H^1}$.

Theorem 1 *under the assumption* (3) *the system* (6) *has unique solution.*

proof. This result is direct consequence of the properties of bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the linear form $F(\cdot)$ (see [3] for more details).

III. The mini-element method

In this section we discretize the Brinkman problem by the finite element, we use the pair $P_1 - Bubble/P_1$ for more detail see [1].

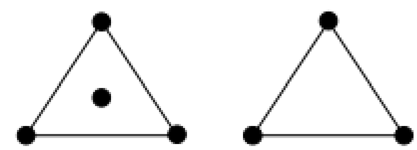

Fig. 1. The mini-element $P_1 - Bubble/P_1$ velocity (left) and pressure (right).

Let T_h be a triangulation of Ω, we denote the discrete space associated with the bubble :

$$B_h = \{v_h \in C(\bar{\Omega}) \; \forall T \in T_h, v_h|_T = xb^T\}, \tag{12}$$

We defined the discrete function spaces :

$$V_{ih} = \{v_h \in C(\bar{\Omega}), v_h|_T \in P_1(T); \; \forall T \in T_h, v_{h/\partial\Omega} = 0\}, \tag{13}$$

for $i = 1, \cdots, d$ and

$$Q_h = \{q_h \in C(\bar{\Omega}) : q_h|_T \in P_1(T); \; \forall T \in T_h, \; \int_\Omega q_h dx = 0\}. \tag{14}$$

where $P_1(T)$ is the set of all 1-order polynomials on triangle T.
Let

$$X_{ih} = V_{ih} \oplus B_h, \tag{15}$$

and

$$X_h = X_{1h} \times X_{2h} \times ... \times X_{dh}. \tag{16}$$

Then, the $P_1 - Bubble/P_1$ finite element approximation of problem is will finding $(u_h, p_h) \in X_h \times Q_h$ such that:

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = F(v_h) & \forall v_h \in X_h, \\ b(q_h, u_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{17}$$

We introduce a set of vector basis functions $\{\phi_i\}$ of the velocity u_h and the pressure p_h, so that :

$$u_h = \sum_{i=1}^{d+1} u_i\phi_i + u_b\phi_b, \qquad p_h = \sum_{i=1}^{d+1} p_i\phi_i,$$

where u_i and p_i are nodal values of u_h and p_h while u_b is the bubble value.

The basis functions in the reference element are defined by [4] :

$$\phi_1(\mathbf{x}) = 1 - x - y, \qquad \phi_2(\mathbf{x}) = x, \qquad \phi_3(\mathbf{x}) = y,$$
$$\phi_b(\mathbf{x}) = 27 \prod_{i=1}^{3} \phi_i(\mathbf{x}).$$

in a two dimensions and

$$\phi_1(\mathbf{x}) = 1 - x - y - z, \qquad \phi_2(\mathbf{x}) = x, \qquad \phi_3(\mathbf{x}) = y,$$
$$\phi_4(\mathbf{x}) = z, \quad \phi_b(\mathbf{x}) = 256 \prod_{i=1}^{4} \phi_i(\mathbf{x}),$$

in a three dimensions.

IV. The linear system

To obtain the numerical solution of the Brinkman equation by mini-element $P_1 - Bubble/P_1$, we replace velocity and pressure approximated equations in the system.
We set :

$$\bar{u_i} = \begin{bmatrix} u_i \\ u_{ib} \end{bmatrix}, \qquad F_i = \begin{bmatrix} f_i \\ f_{ib} \end{bmatrix}, \quad i = 1, \cdots, d.$$

Using (17), we obtain the following algebraic form

$$\mathcal{K}\mathcal{U} = \mathcal{B}. \tag{18}$$

Where

$$\mathcal{K} = \begin{bmatrix} K & 0 & B_1^t \\ 0 & K & B_2^t \\ B_1 & B_2 & 0 \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix},$$

in $2D$ and

$$\mathcal{K} = \begin{bmatrix} K & 0 & 0 & B_1^t \\ 0 & K & 0 & B_2^t \\ 0 & 0 & K & B_3^t \\ B_1 & B_2 & B_3 & 0 \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{bmatrix},$$

in $3D$.

Where $K = M + S$, with M and S are the mass matrix and stiffness matrix respectively, B_i is the divergence submatrix,

$$B_i = -(q_i, \partial_i u_{ih})_\Omega \ , \qquad i = 1, \cdots, d.$$

and we define the element matrices and vectors:

$$\bar{M}_{ij} = \int_T \tilde{\mu} \nabla\phi_i \nabla\phi_j dx, \quad \bar{S}_{ij} = \int_T K^{-1} \mu \phi_i \phi_j dx,$$
$$\bar{B}_{ij} = -\int_T \partial_1 \phi_i \phi_j dx - \int_T \partial_2 \phi_i \phi_j dx,$$
$$\bar{f}_i = \int_T f \phi_i dx.$$

Assembling the element matrices over the triangulation T_h, we obtain the global matrices

$$M = (M_{ij}), \quad S = (S_{ij}), \quad B = (B_{ij}), \quad F_i = (f_i).$$

Where,

$$M_{ij} = \sum_T \bar{M}_{ij}, \quad B_{ij} = \sum_T \bar{B}_{ij},$$
$$S_{ij} = \sum_T \bar{S}_{ij}, \quad f_i = \sum_T \bar{f}_i.$$

To solve the big system it's possible to use an iterative solver for the global linear system, in this study we use Uzawa conjugate gradient Algorithm, this solver applied for the linear system:

$$\begin{bmatrix} A & B^t \\ B & -C \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F \\ F_b \end{bmatrix}, \tag{19}$$

where C is symmetric positive semi-definite and A is symmetric positive definite block diagonal matrix. The augmented Lagrangian operator define by:

$$L(U, P) = \frac{1}{2} U^t A U - F^t U + P^t B U - \frac{1}{2} P^t C P - F_b P, \tag{20}$$

since A is symmetric positive definite and C is symmetric positive semi-definite, The saddle-point for operator (20) exists. As result the system (19) characterizes the solution of the saddle-point problem:

$$\max_U \min_P L(U, P) = \min_U \max_P L(U, P). \tag{21}$$

The Uzawa's algorithm is simple algorithm for resolute our problem, which we write here as:
We consider U_P solution for the Poisson problem:

$$AU = F - B^t P, \tag{22}$$

by using the two formulas (20) and (22) we obtain

$$L(U_P, P) = -\frac{1}{2} U^t A U - \frac{1}{2} P^t C P - F_b^t P. \tag{23}$$

We denote

$$\mathcal{L}^*(P) = -\min_U L(U_P, P), \tag{24}$$

Therefore, the problem (21) comes down to finding P such that

$$\mathcal{L}^*(P) \le \mathcal{L}^*(q) \qquad \forall q \in Q, \tag{25}$$

by using (23) the $\mathcal{L}^*$ is a quadratic and coercive, with X is the solution of the sensitivity problem

$$AX = B^t D, \tag{26}$$

From (22) the function $U \mapsto U_P$ is a linear and

$$U_{P+TD} = U_P + TX, \tag{27}$$

with X indicate in (26).
Therefore,

$$\nabla \mathcal{L}^*(P) = BU + CP + F_b. \tag{28}$$

Under direction of D, we will calculate an optimal stepsize φ such that

$$\nabla \mathcal{L}^*(P + \psi D)^t D = 0, \tag{29}$$

by using $k := \nabla \mathcal{L}^*(P)$ the ψ define by:

$$\psi = \frac{-D^t(BX + CD)}{k^t D}. \tag{30}$$

The Fletcher-Reeves conjugate gradient direction at each iteration i is given by

$$D_i = k_{i+1} + \alpha_i D_i, \tag{31}$$

$$\alpha_i = \| k_i \|^{-2} \| k_{i+1} \|^2 . \tag{32}$$

For more details about the Uzawa conjugate gradient see [13], [14] and the script algorithm see [4].

V. The numerical experiments

In this section, some numerical results of calculations with the mini-element $P_1 - Bubble/P_1$ will be presented. We consider a test problem present below [5], our numeric experience is summarized in the figure 2. In this simulation we take the

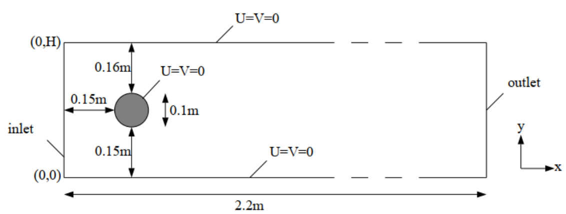

Fig. 2. Geometry and the boundary of the flow arround a cylinder with circular cross section in $2D$.

permeability tensor is a matrix defined by:

$$\mathbf{K^{-1}} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \tag{33}$$

with α_1 and α_2 are two positive reals. This matrix satisfy the condition (3). Using MATLAB software and change the value of the component of permeability matrix. We take in this example with the Newtonian fluid vescosity and the physical dynamic viscosity equals one (i.e $\mu = \tilde{\mu} = 1$).

Remark: If $\mathbf{K} \to \infty$ then equations (1) turn to be the classic Stokes problem, and our problem coincides to the classical Darcy's problem for $\tilde{\mu}$ tends to 0.

We consider the homogeneous boundary conditions of Ω except:

In the left boundary we give

$$u_1 = \frac{0.3}{0.41^2} \times 4y(0.41 - y); \quad u_2 = 0, \tag{34}$$

and in the outflow boundary we give

$$u_1 = \frac{0.3}{0.41^2} \times 4y(0.41 - y); \quad u_2 = 0 \tag{35}$$

If the domain Ω is bounded and simply connected, in order to compute the stream-function φ, we have to solve the Poisson Neumann problem

$$-\Delta\varphi = \varpi \quad in \quad \Omega, \tag{36}$$

$$\partial_n \varphi = -u.\tau, \tag{37}$$

where $\varpi = \partial_1 u_2 - \partial_2 u_1$ is the vorticity and τ the counter-clockwise oriented unit tangent vector at $\partial\Omega$. The variational formulation of (36)-(37) is find $\varphi \in H^1(\Omega)$ such that:

$$\langle \nabla\varphi, \nabla\phi \rangle = \langle u_1, \partial_2\phi \rangle - \langle u_2, \partial_1\phi \rangle, \quad \forall \phi \in H^1(\Omega). \tag{38}$$

Problem (36)-(37) has a unique solution in $H^1(\Omega)$.

By using the P_1 finite element, we obtain the algebraic system:

$$\mathbf{R}\varphi = \mathbf{B}_2^t u_1 + \mathbf{B}_1^t u_2, \tag{39}$$

where $\mathbf{R}$ is the two dimensional Laplacian matrix. To ensure the uniqueness for the problem (39), we impose the pressure φ equal zero at an arbitrary node.

Under the conditions declared previously, here our different simulation:

Velocity: The Figures (3)–(4) present the velocity vectors in the different cases: $\alpha_1 = \alpha_2 = 1$, next $\alpha_1 = 1$ and $\alpha_2 = 10^{-4}$.

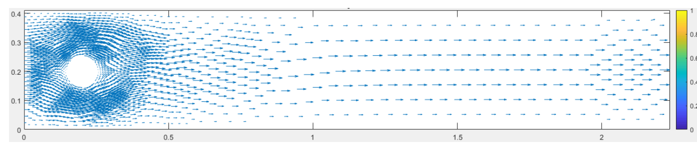

Fig. 3. Velocity field, $\alpha_1 = \alpha_2 = 1$.

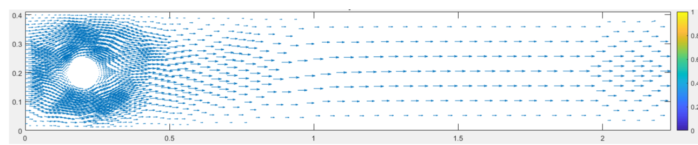

Fig. 4. Velocity field, $\alpha_1 = 1$ and $\alpha_2 = 10^{-4}$.

Isobar: The Figures (5)–(6) present the isobar lines in the different cases: $\alpha_1 = \alpha_2 = 1$, next $\alpha_1 = 1$ and $\alpha_2 = 10^{-4}$.

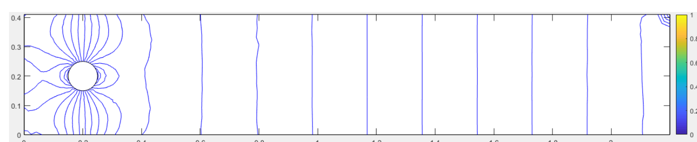

Fig. 5. Isobar lines, $\alpha_1 = \alpha_2 = 1$.

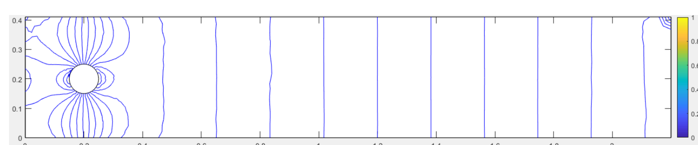

Fig. 6. Isobar lines, $\alpha_1 = 1$ and $\alpha_2 = 10^{-4}$.

Streamlines: The Figures (7)–(8) present the streamlines in the different cases: $\alpha_1 = \alpha_2 = 1$, next $\alpha_1 = 1$ and $\alpha_2 = 10^{-4}$. The streamlines are obtained by plotting the solution of the variational problem (38).

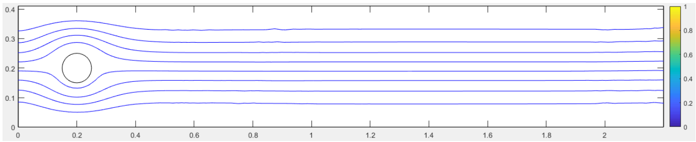

Fig. 7. Streamlines, $\alpha_1 = \alpha_2 = 1$.

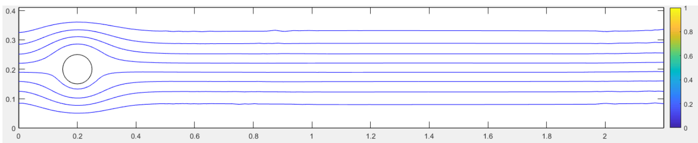

Fig. 8. Streamlines, $\alpha_1 = 1$ and $\alpha_2 = 10^{-4}$.

Conclusion

We were interested in this work on the numeric solution of Brinkman's equations in a heterogeneous medium media porous. In this study, we used discretization of mini-element methods $P_1 - Bubble/P_1$. We use a Uzawa conjugate gradient method to accelerate the approach solution for the linear system. The numeric solution has shown the accuracy and efficiency of the proposed finite element method. This method is very important to study a more complex problem, for example very highly heterogeneous porous media.

Acknowledgment

The authors would like to express their sincere thanks for the referee for his/her helpful suggestions.

References

[1] D. Arnold, F. Brezzi and M. Fortin, "A stable finite element for the Stokes equations," Calcolo, vol. 21, pp. 337-344, 1984.

[2] D. Boffi, and F. Brezzi, and M. Fortin, and others, "Mixed finite element methods and applications," Heidelberg Springer, vol. 44, 2013.

[3] F. Brezzi, "On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers," RAIRO Anal.Num., vol. 8, pp. 129-151, 1974.

[4] J. Koko, "Efficient MATLAB codes for the $2D/3D$ Stokes equation with the mini-element," Informatica, vol. 30, pp. 243-268, 2019.

[5] M. Schafer, and S. Turek, "Benchmark computations of laminar flow around a cylinder," Notes on Numerical Fluid Mechanics, vol. 52, pp. 547–566, 1996.

[6] M. Juntunen, and R. Stenberg, "Analysis of finite element methods for the Brinkman problem," Springer, Calcolo, vol. 47, pp. 129–147, 2010.

[7] J. El-Mekkaoui, A. Elkhalfi, A. Elakkad, "Resolution of Stokes Equations with the $C_{a,b}$ Boundary Condition Using Mixed Finite Element Method," WSEAS Transactions on Mathematics, vol. 12, pp. 586–597, 2013.

[8] O. El Moutea, and H. El Amri, and A. El Akkad, "Mixed finite element method for flow of fluid in complex porous media with a new boundary condition," Computer Science, vol. 15, pp. 413–431, 2020.

[9] J. Volker, "Finite Element Methods for Incompressible Flow Problems," Springer International Publishing, vol. 51, 2016.

[10] J. Koplik, H. Levine, and A. Zee, "Viscosity renormalization in the Brinkman equation," The Physics of fluids, vol. 26, no. 10, pp. 2864-2870, 1983.

[11] P. A. Raviart, J. Thomas, "Introduction à l'analyse numerique des equations aux dérivées partielles," Masson, Paris, 1983.

[12] A. Elakkad, A. Elkhalfi, N. Guessous, "An a posteriori error estimate for mixed finite element approximations of the Navier-Stokes equations," Journal of the Korean Mathematical Society, vol. 48, no. 3, pp. 529-550, 2011.

[13] H. James, E. Joseph, and T. Apostol, "Analysis of the inexact Uzawa algorithm for saddle point problems," SIAM Journal on Numerical Analysis, vol. 34, no. 3, pp. 1072-1092, 1997.

[14] W. Hackbusch, "Iterative solution of large sparse systems of equations," Springer, vol. 95, pp. 210-262, 1994.

[15] F. Brezzi, M. Fortin, "Mixed and Hybrid Finite Element Methods," Springer-Vedag, New York, 1991.

[16] V. Giraut, P. A. Raviart, "Finite Element Methodsfor Navier-Stokes Equations," Springer-Verlag, New York, 1986.

[17] A. Ern, "Aide-Memoire Elements Finis," Dunod, Paris, 2005.

[18] P. Angot, "Analysis of singular perturbations on the Brinkman problem for fictitious domain models of viscous flows," Mathematical methods in the applied sciences, vol. 22, no 16, pp. 1395-1412, 1999.

[19] O. Iliev, R. Lazarov, J. Willems, "Variational multiscale finite element method for flows in highly porous media," Multiscale Model. Simul., vol. 9, no. 4, pp. 1350-1372, 2011.

[20] Y. Mao, "Geometric multigrid methods for flow problems in highly heterogeneous porous media," (Ph. D. thesis). Texas A.M University, College Station, 2014.

[21] P. Hansbo and M. Juntunen, "Weakly imposed Dirichlet boundary conditions for the Brinkman model of porous media flow," Applied Numerical Mathematics, vol. 59, no. 6, pp. 1274–1289, 2009.

[22] K. A. Mardal, X.-C. Tai, and R. Winther, "A robust finite element method for Dacry-Stoke flow," SIAM Journal on Numerical Analysis, vol. 40, no. 5, pp. 1605–1631, 2002.

[23] G. Kanschat, R. Lazarov, and Y. Mao, "Geometric multigrid for darcy and brinkman model of flows in highly heterogeneous porous media: a numerical study," Journal of Computational and Applied Mathematics, vol. 310, pp. 174–185, 2017.

[24] Q. Zhai, R. Zhang, and L. Mu, "A new weak Galerkin finite element scheme for the Brinkman model," Communications in Computational Physics, vol. 19, no. 5, pp. 1409–1434, 2016.

[25] A. Hannukainen, M. Juntunen, and R. Stenberg, "Computations with finite element methods for the Brinkman problem," Computational Geosciences, vol. 15, no. 1, pp. 155–166, 2011.